OPEN-ENDED PROBLEM SOLVING
Weaving a Web of Ideas

STORY BY Denise Jarrett PHOTOS BY Suzie Boss

THE ALLURE OF PROBLEM solving isn't just for great mathematicians. Being absorbed in a difficult task is common to all of us, and, often, these moments prove to be some of our most rewarding experiences.

Take the case of Andrew Wiles. At 33, the Princeton University professor set out to prove Fermat's Last Theorem, joining a 350-year-old contest. Pierre de Fermat posed this challenge to the world in the 17th century by writing a note in the margin of his favorite book of ancient mathematics. He wrote that he'd developed a proof for a particularly knotty problem, but there wasn't enough room in the margin to write it all out.

Fermat's theorem stated that there are no whole number solutions for *n* greater than 2 for the equation $x^n + y^n = z^n$ (an extrapolation of the Pythagorean Theorem, $x^2 + y^2 = z^2$).

In the centuries since Fermat provoked the contest, legions of mathematicians have tried to build on the scanty hints he left behind. One of the first great breakthroughs came in the 19th century by French mathematician Sophie Germain. Like Wiles, she was transfixed by the problem, working on it single-mindedly for several years. Germain's work paved the way for more breakthroughs. But it remained for Wiles to solve at the end of the 20th century, drawing on all the power of modern mathematics (Singh, 1997).

The story of Wiles' pursuit of the elusive proof is dramatic and emotional. He worked on it for eight years, experiencing all manner of ups and downs. His struggle to find a solution epitomizes the power and allure of open-ended problem solving. His triumph in 1993-94 made for exultant headlines and a riveting BBC Horizon documentary film—the theorem even found its way into *Arcadia*, Tom Stoppard's play of that year about love, mathematics, and the nature of scientific discovery.

Young problem solvers buddy up at Clarendon Elementary School in Tacoma, Washington.

Wiles' story appeals, even to those who don't see themselves as mathematically inclined, because solving problems is a basic human drive. We may not understand the mathematics involved in Wiles' proof, indeed few mathematicians do, but we understand his enormous capacity for curiosity, perseverance, and resiliency.

We understand, perhaps, because we've all experienced that state of "flow," as psychologist Mihaly Csikszentmihalyi (say Chick-SENT-me-high) terms it, in which we're so focused on doing something that we lose track of time. These moments of immersion in a meaningful challenge are not only some of our most satisfying experiences in life, they are also the richest for learning.

The concept of flow is a useful analogy to describe the level of engagement Wiles must have experienced while grappling with Fermat's problem. This level of absorption is the kind of learning experience students are meant to attain through open-ended problem solving.

Czikszentmihalyi (1990) writes that flow occurs when a person's abilities are fully engaged in overcoming a challenge that is interesting and "just about manageable." During flow, the person is controlling the direction of his approach to the task, constructing his own learning as he stretches his abilities to master the activity, whether it be building a shed, quilting, writing a poem, or solving a math problem.

Freedom to make one's own decisions about how to approach a problem and what strategies to employ is also a key to open-ended problem solving. This requires a fundamental shift away from traditional methods that emphasize teacher-directed rote learning.

While the phrase *open-ended problem solving* may sound forbidding, it basically describes this heightened learning experience of being fully absorbed in a difficult and interesting task. Research shows open-ended problem solving to be particularly effective in promoting deep mathematical understanding (Hiebert, Carpenter, Fennema et al., 1996; Schoenfeld, 1992). A teacher's role is to make classroom conditions favorable for this kind of learning to take place.

weaving ideas

Czikszentmihalyi notes that "playing with ideas is extremely exhilarating" and when made a lifelong habit, it weaves a web of connected ideas, giving resiliency and snap to intelligence. Transfer this concept to the mathematics classroom, and the writings of James Hiebert, Professor of Education at University of Delaware, come to mind. Hiebert's research on problem solving has done much in the past 20 years to help identify and articulate the features and benefits of problem solving in the teaching and learning of mathematics. The real value of problem solving, he concludes, is in the ideas it produces.

When a student learns math by grappling with difficult and absorbing problems—rather than by simply memorizing and practicing predetermined procedures—she is free to "wonder why things are, to inquire, to search for solutions, and to resolve incongruities," he says. "This approach yields deep understandings of the kinds that we value" (Hiebert, Carpenter, Fennema, et al., 1996).

> "THINGS LEARNED WITH UNDERSTANDING ARE THE MOST USEFUL THINGS TO KNOW IN A CHANGING AND UNPREDICTABLE WORLD."
> **—JAMES HIEBERT AND COLLEAGUES (1996)**

Thinking abstractly about ideas increases the flexibility of one's thinking capacity. (Though ideas within a "real-world" context are also valuable.) The National Research Council explains the importance of developing flexible thinking in its influential 1989 publication *Everybody Counts: A Report to the Nation on the Future of Mathematics Education*:

"Experience with mathematical modes of thought builds mathematical power—a capacity of mind of increasing value in this technological age that enables one to read critically, to identify fallacies, to detect bias, to assess risk, and to suggest alternatives."

To help young people to be better problem solvers is to prepare them not only to think mathematically, but to approach life's ever-changing challenges with confidence in their problem-solving ability.

open-ended problem solving

Open-ended problem solving involves problems that have multiple solution methods and answers. Teachers should choose problems that are just beyond the solver's skill level. The difficulty should be an intellectual impasse, notes Schoenfeld (1992), rather than a computational one. In fact, students who haven't yet mastered computations should be allowed to do open-ended problem solving. Nonroutine problems can provide ample opportunity to build computation skills while engaging the student in more challenging mathematics and higher-order thinking. All students deserve the opportunity to develop their problem-solving ability.

Working collaboratively is a key feature of open-ended problem solving, though students will also work individually. To solve a problem, students draw on their previous knowledge and experience with related problems. They decide which solution method to follow, perhaps even constructing their own method, trying this and that, before arriving at a solution. They then reflect on the experience, tracing their own thinking processes and reviewing the strategies they attempted, determining why some worked and others didn't. Students will discuss the problem, identifying its features, considering possible solution methods, conjecturing, and explaining their thinking.

"Communication works together with reflection to produce new relationships and connections," writes Hiebert (1996). "Students who reflect on what they do and communicate with others about it are in the best position to build useful connections in mathematics."

(Even Andrew Wiles, who worked in seclusion most of those eight years, needed the direct collaboration of others to help him finally solve Fermat's Last Theorem.)

Through it all, the solver constructs a mathematical understanding of the problem that is both deep and flexible. By connecting her prior knowledge with new concepts and skills, she gains depth of understanding. These connections also create flexibility in her thinking, enabling her to extend her knowledge to new mathematical situations and beyond. Good problem solvers, says Robert McIntosh, Mathematics Associate for the Northwest Regional Educational Laboratory, can see past the surface features of problems to common underlying structures. They can monitor their own thinking strategies, recognizing when an approach or tactic is not being productive and modifying it as necessary. Self-awareness and the ability to reflect are essential for improving one's problem-solving ability. Furthermore, good problem solvers are resourceful, confident, and willing to explore. They're persistent and tolerate a measure of frustration.

Japanese students are some of the most tenacious problem solvers. Indeed, when the U.S. members of the Third International Mathematics and Science Study viewed videotapes of Japanese math classes, they were amazed by students' perseverance (Stigler & Hiebert, 1999). More pointedly, another comparative study of first-grade students working a difficult task (in fact, the task was unsolvable, but the students didn't know that), reported that U.S. students gave up in about 15 seconds, while Japanese students didn't stop until the class came to an end an hour later (Stigler, 1999).

"To develop these abilities, students need ample opportunities to experience the frustration and exhilaration that comes from struggling with, and overcoming, a daunting intellectual obstacle," says McIntosh.

The beauty and utility of open-ended problem solving is just this: It leads to understanding that is transferable. And in this increasingly complex world, the ability to transfer knowledge and skills to meet changing conditions and challenges is essential.

standards: a statement of values

Standards, says Hiebert (1999), are simply a statement about what we most value. From our best judgment, we create education standards based on past experiences, research, advice from practitioners, and societal expectations. At least since the 1940s when George Pólya identified problem solving as an essential math skill—the *heart* of mathematics, in fact—problem solving has been a stated education priority. When the National Council of Teachers of Mathematics began issuing its standards for math teaching and learning in the 1980s, problem solving rose to the fore of standards reform. In its *Principles and Standards for School Mathematics (NCTM, 2000)*, the council continues to identify problem solving as a core strand of math learning for all grade levels.

Manipulatives help these students solve a geometric problem.

Across the country, states are writing mathematics standards and assessments to include problem solving, acknowledging the importance of assessing students' ability to reason, communicate, make connections, and apply their knowledge to problem situations. Thus, problem solving is an area teachers are increasingly expected to teach and assess.

making the change

Teachers are often caught between daily pressure from colleagues, parents, and community members to uphold convention in the classroom, and pressure from administrators and policymakers to employ standards-based practices that show immediate and positive results on achievement tests. One must consider these opinions and mandates, but teachers who make meaningful changes are those who develop their own inner voice of authority (Wilson & Lloyd, 2000).

Teaching problem solving is an art mastered over a long period of time (Thompson, 1989). By reflecting on their personal understandings of teaching and learning—as well as their students' understandings—teachers develop inner authority for improving their instruction. Through reflection, teachers determine for themselves the value of particular reform innovations (Wilson & Lloyd, 2000).

Many teachers do recognize that nontraditional strategies are necessary to meet the learning needs of their increasingly diverse students. Embracing change can be unsettling, but these teachers venture into new territory, opening a world of discovery for themselves and their students. For they know that a young mind carefully nurtured may be the next big thinker to solve another of the world's mysteries. NWT

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NONROUTINE PROBLEMS

If problem solving is at the heart of mathematics, then nonroutine problems are at the heart of problem solving.

True problem solving involves nonroutine, or open-ended, problems. Moving into the territory of nonroutine problems is full of unknowns. Solution methods and answers are not made explicit. Decisionmaking is shared between teachers and students. Teachers must predict what tactics and questions might come up in class, preparing for them as best they can.

Teachers need mathematical expertise to anticipate students' approaches to a problem and how promising those approaches might be. They must choose tasks which are appropriately difficult for their particular students. They must decide when and how to give help so that students can be successful but still retain ownership of the solution. Teachers will sometimes find themselves in the uncomfortable position of not knowing the solution to a problem. Letting go of the "expert" role requires experience, confidence, and self-awareness.

Teachers should choose tasks with nonroutine problems that (Hiebert et al., 1996):

- Make the subject problematic so students see the task as interesting and challenging
- Connect with students' present level of understanding, allowing them to use their knowledge and skills to develop methods for completing the task
- Allow students to reflect on important math ideas